a guide to the color film 23min

Turning a Sphere Inside Out

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INTRODUCTION

TURNING A SPHERE INSIDE OUT opens with a discussion of the problem of turning a sphere inside out by passing the surface through itself without making any holes or creases. Mathematicians believed that the problem was insoluble until 1958 when Stephen Smale proved otherwise. However, no one could visualize the motion, called a regular homotopy. The homotopy in this film was developed by Bernard Morin, a blind mathematician. It is illustrated with a sequence of chicken-wire models, built by Charles Pugh, showing the crucial stages in the motion. Mathematicians Nelson L. Max, Stephen Smale, and Charles Pugh, and physicist Judith Bregmann provide the commentary. The film closes with several different sequences of computer animation revealing the continuous motion of the sphere.

Crucial stages in this motion are given in the discussion following, with figures placed above the text referring to them. The inner surface of the sphere has been shaded.

TURNING A SPHERE INSIDE OUT

Imagine that the surface of the sphere is made out of rubber like a hollow ball. If we make a hole in the ball, we can turn it inside out through the hole, but we have destroyed the surface. At the intermediate stages, it is no longer a complete sphere.

If we do not allow any holes, the problem becomes impossible; what is inside must remain inside. Therefore, instead of allowing holes, we allow the surface to cross itself. A physical sheet of rubber can never cross itself, but a mathematical surface in space can—just as a curve drawn on a piece of paper can cross itself. A position of the surface is merely a function from a round sphere into three-dimensional space; if the surface crosses itself, this only means that the function is not one-to-one.

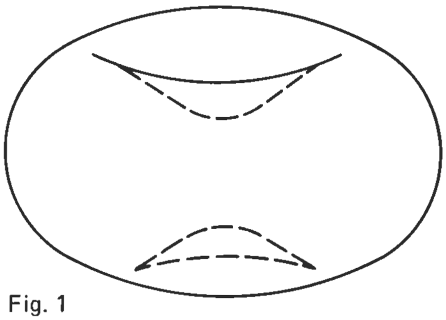

Fig. 1

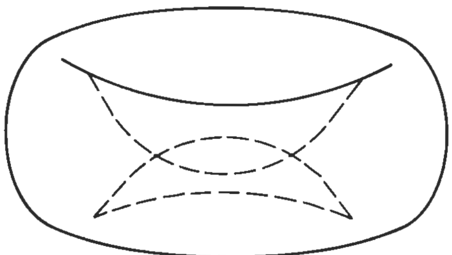

Fig. 2

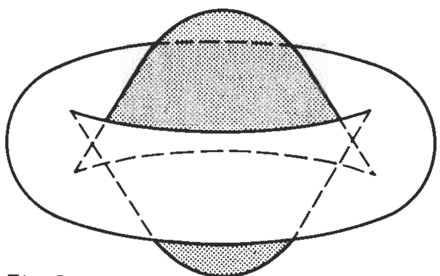

Fig. 3

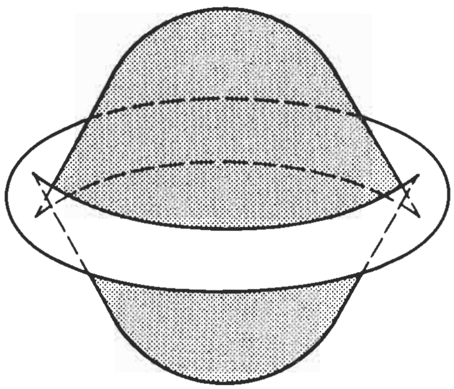

Fig. 4

The north pole can be pushed down past the south pole so that the sphere intersects itself along a circle and the inside of the surface near the south pole becomes visible from above. As the surfaces are pushed through each other, the circle of intersection grows.

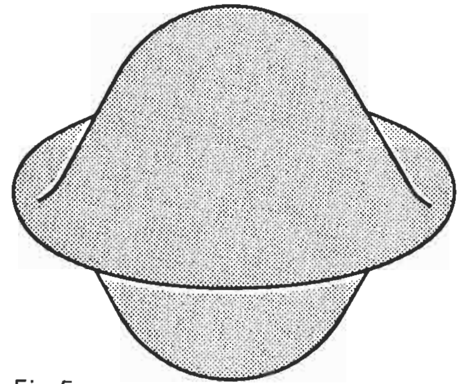

Fig. 5

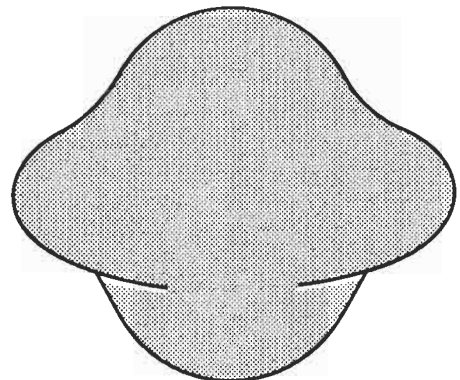

Fig. 6

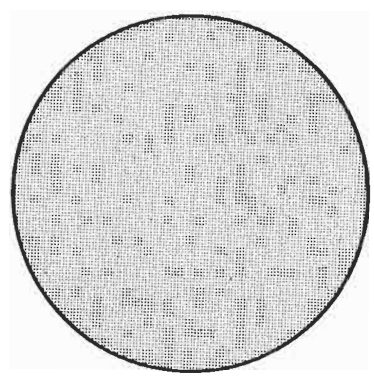

Fig. 7

Eventually the intersection circle reaches the equator and forms a crease there. Then the surface opens up into a round inside out sphere.

The crease is disagreeable because the surface is not entirely smooth; the function defining the surface is not differentiable. So we add another rule stating that every surface during the motion must be smooth. Mathematically this means that the partial derivative vectors must be continuous and linearly independent. Geometrically it means there is a tangent plane at every point on the surface. There is no tangent plane to a surface at a crease.

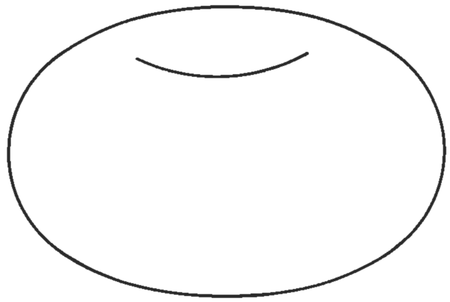

Fig. 8

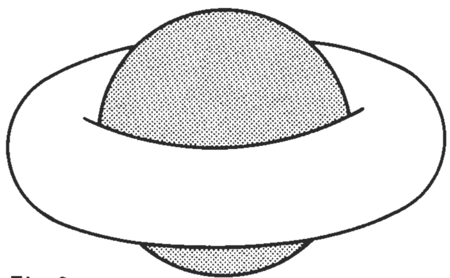

Fig. 9

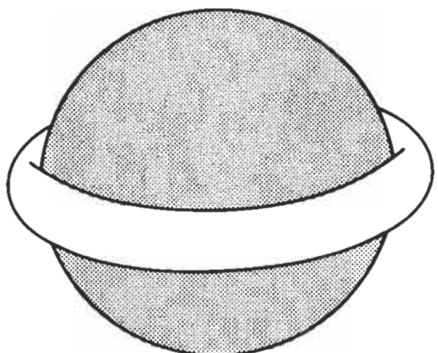

Fig. 10

There is a tricky way around this rule. Near the equator we make a smaller and smaller round tube which disappears at the last instant. At every stage we have a smooth surface, even when the tube disappears in a "kink." This is the three dimensional version of the method of turning a figure eight into a circle by making a little loop disappear, as shown in REGULAR HOMOTOPIES IN THE PLANE: PART I. We prohibit this too.

A motion which satisfies all the rules—no holes, creases, or "kinks"—is called a regular homotopy. Our problem is to describe a regular homotopy which turns the sphere inside out.

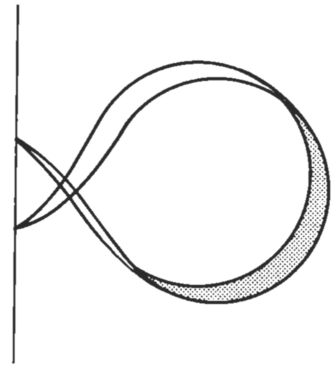

Fig. 11

The motions we have described so far all involved surfaces of revolution, obtained by revolving half of a cross section curve around a vertical axis. None gave regular homotopies, since each violated one of the rules.

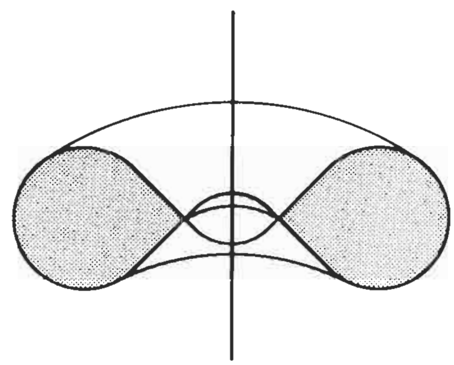

Fig. 12

Is there a regular homotopy which turns the sphere inside out through surfaces of revolution? If there were, each cross section through the axis would be a smooth curve in the cross section plane, and we would have a regular homotopy in the plane which turns a circle inside out.

The Whitney-Graustein theorem, discussed in the films REGULAR HOMOTOPIES IN THE PLANE: PARTS I and II, proves that the circle cannot be turned inside out by a regular homotopy. Therefore, the sphere cannot be turned inside out in space by a regular homotopy using only surfaces of revolution. This fact led mathematicians to believe that it was impossible to turn a sphere inside out by any regular homotopy at all. However, in 1958, Stephen Smale proved that in fact it was possible. His proof used a complicated sort of mathematical induction which pushed the surface wildly back and forth at each step. Although the construction of a regular homotopy was, in principle, contained in his proof, no one could visualize it.

Once they knew it was possible, a number of people invented homotopies; among them was Arnold Shapiro, René Thom, Anthony Phillips, Marcel Froissart and Bernard Morin. Anthony Phillips, wrote an article in the May, 1966 issue of *Scientific American* giving a history of the problem and a sequence of illustrations for the regular homotopy. The one illustrated here is somewhat simpler and was invented by Bernard Morin, a blind French mathematician who described it using the ideas of generic surfaces and singularities.

GENERIC AND SINGULAR POSITIONS

A generic surface is one which will look basically the same if it is moved slightly. For example, if a surface intersects itself in a circle, and we move it to a slightly different position, it will still intersect itself in a nearby curve, which may not be exactly circular. If we imagine the surface imbedded in solid rubber or gelatin, we could move the gelatin from one position to the other and carry the surface along. Two positions of a surface are called topologically equivalent if they can be moved one to the other by such a pushing or twisting of all of space. So a position of a surface is called generic if all small motions give nearby surfaces which are topologically equivalent to it.

For example, the positions shown in Figures 2, 3, and 4 are all generic, as are all the stages between them. As a consequence, the positions in Figures 2 and 4 are topologically equivalent. Similarly all the positions before the north and south poles touch are topologically equivalent.

However, at the instant the north and south poles touch, the topology changes. If we pull the north and south poles apart, there will be no self-intersection, and if we push them past each other, there will be a circle of self-intersection. These are not equivalent to the position with one point of tangency, which is therefore not generic. We call the point of tangency a singular point, the position a singular position or singularity.

In a regular homotopy, the singular positions will in general occur at a finite number of special instants, at which the nature of the self-intersection changes. Between these instants, the surfaces are merely pushed or twisted by topological equivalences. Thus, a regular homotopy can be described in terms of a list of its singular positions, such as the one given at the end of these notes. Before we start on this list, let us look at standard models for the four types of singular points which occur in it.

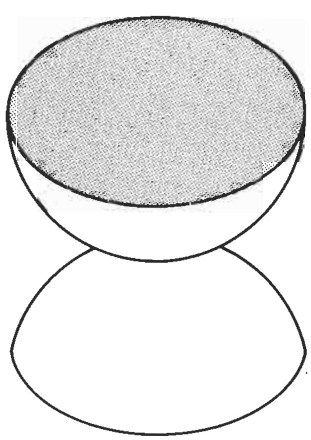

Fig. 13

We have already seen one type, a tangency at which a circle of self-intersection is created. A standard model for this is two curved bowls, tangent at one point.

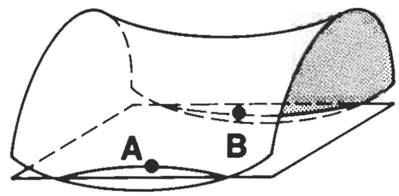

Fig. 14

The second sort of singular point occurs when an isthmus of land is flooded by a rising tide. In the standard model above, the isthmus is represented by the saddle shaped surface, and the sea level by the horizontal plane. Points A and B are on the shoreline of the isthmus at its narrowest part.

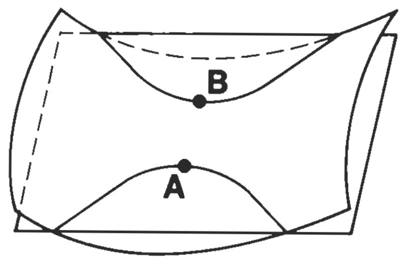

Fig. 15

Here is a view from the top, showing the shoreline as two arcs passing from left to right through the points A and B.

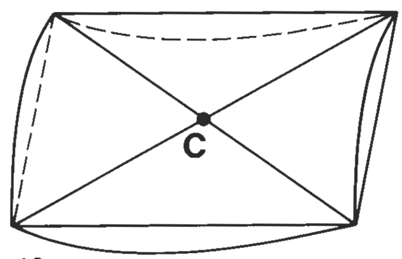

Fig. 16

When the sea rises to the low point, C, of the pass on the isthmus, it is tangent to the saddle shaped surface there. The two shorelines through A and B have met at C. This position is not topologically equivalent to those just preceeding it, so it is singular, and C is the singular point.

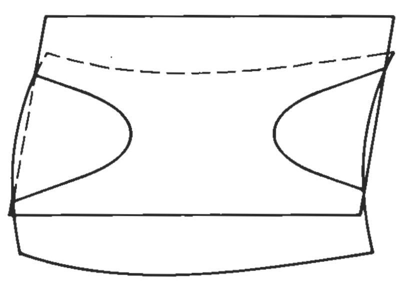

Fig. 17

As the water level rises still more, the two regions of land are separated by a shallow channel of water. There are again two arcs of shoreline representing the surface intersections, but they connect differently, and now run from top to bottom.

For our next singularity, we consider a triple point where three surfaces intersect. If the surfaces intersect like the two walls and ceiling at the corner of a room, then the triple point is not singular, because the surfaces would intersect in a topologially equivalent way if they were moved slightly.

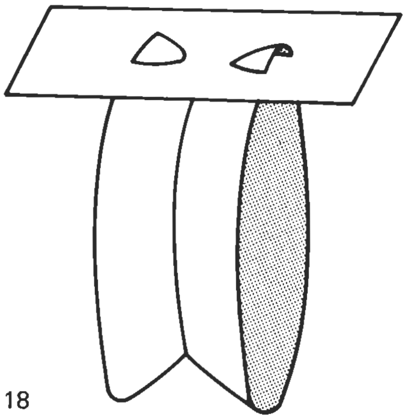

Fig. 18

But consider the picture above, showing two bowls with a vertical intersection circle, and a horizontal plane just above this circle.

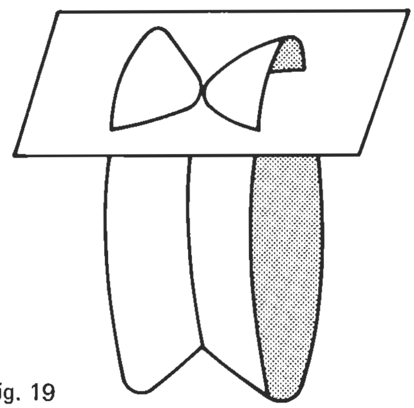

Fig. 19

If this plane is lowered until it is tangent to this circle, we get a singular triple point, where three intersection curves are tangent.

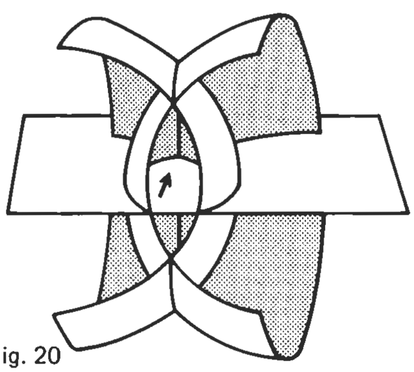

Fig. 20

If the plane is lowered still more, this singular triple point separates into two non-singular triple points, where the three self-intersection curves cross. One of them is shown at the arrow in the cut away picture above.

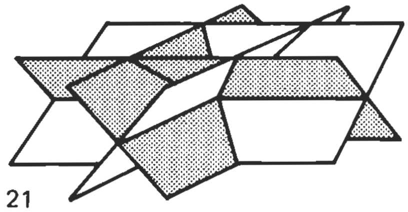

Fig. 21

The final sort of singularity is represented by a quadruple point, where four surfaces intersect, as in the picture above.

This quadruple point is singular, since if any of the planes were moved a little, it would separate into four triple points, forming the vertices of a small tetrahedron.

MORIN'S REGULAR HOMOTOPY

Now we are ready to study Morin's regular homotopy in terms of these singularities. Here are the first few stages, shown in cross section.

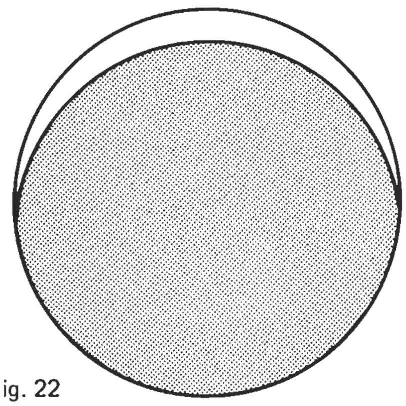

Fig. 22

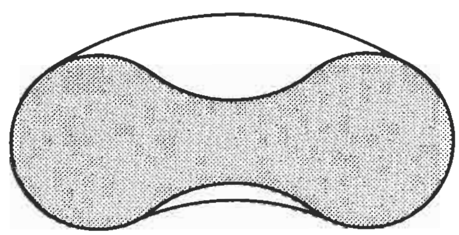

Fig. 23

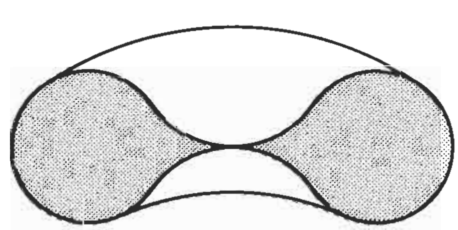

Fig. 24

The first singular point occurs when the north and south poles touch, and has Figure 13 as its standard model.

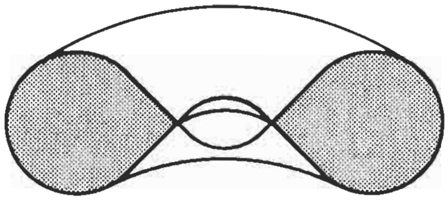

Fig. 25

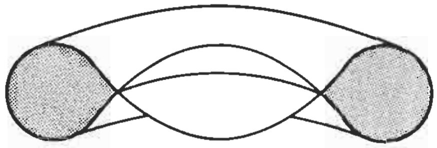

Fig. 26

A horizontal self-intersection curve is created at this singular point, and then grows larger.

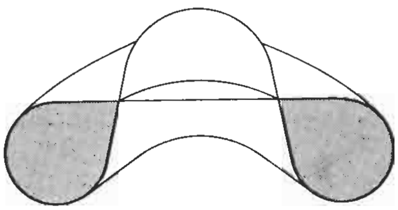

Fig. 27

The next stage in the homotopy involves twisting the surface so that it looks like a hat,

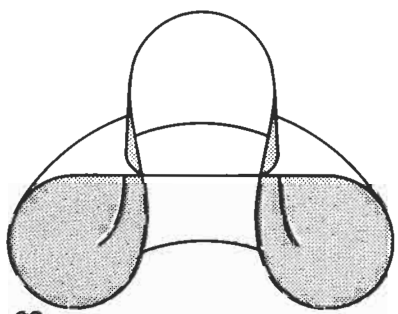

Fig. 28

and bringing the sides inward

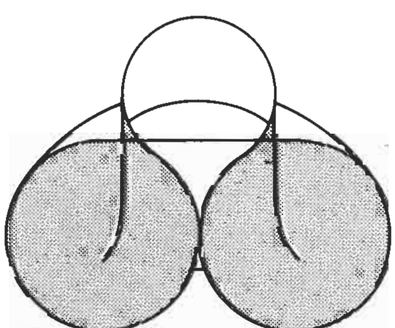

Fig. 29

so that they touch,

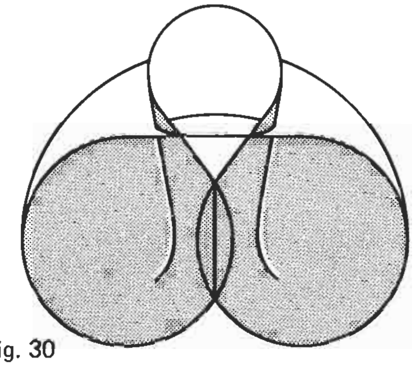

Fig. 30

and pass through each other, forming a vertical circle of self intersection. This is the same sort of singular point, represented by the model in Figure 13 turned sideways.

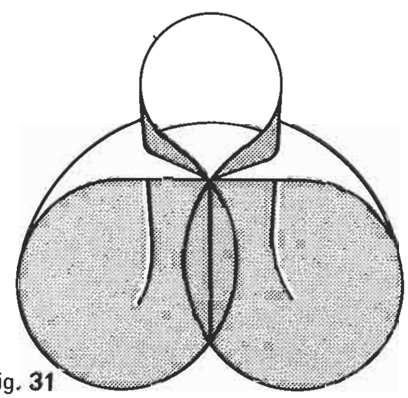

Fig. 31

In the next few stages, the second vertical circle of self-intersection continues to expand as the sides are pushed in. It soon touches the horizontal plane at a singular triple point like the one in Figure 19.

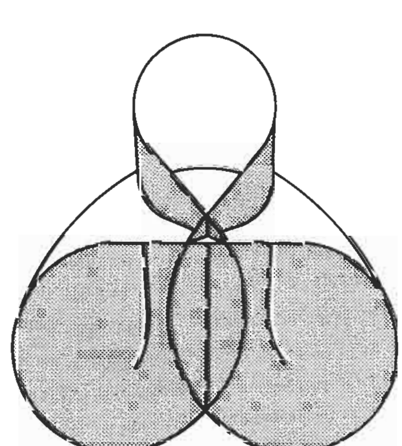

Fig. 32

Pushing in the sides has the same effect as lowering the horizontal plane: Two non-singular triple points are created, as in Figure 20.

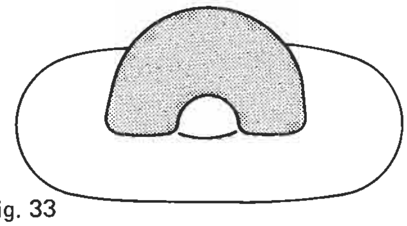

Fig. 33

Here is a perspective view of the surface so far, viewed from the side.

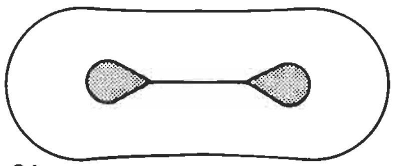

Fig. 34

Here is the same surface viewed from below. The two shaded areas are part of the back of the horizontal surface. We call them "eardrums" and call the tubes around them "ears" or "earlobes." There are two mirror planes of reflection symmetry, perpendicular to the page in this view. The surface is cut by them into four symmetrical quarters. The horizontal segment shows an arc of the "vertical" intersection circle.

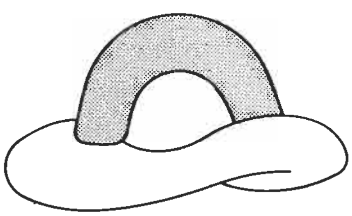

Fig. 35

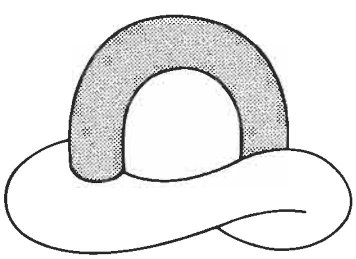

Fig. 36

The next stage involves twisting of the surface. The tube across the top grows longer, and the horizontal section becomes twisted so that the two "ears" on the sides lie in different planes. The twisted shapes are all topologically equivalent.

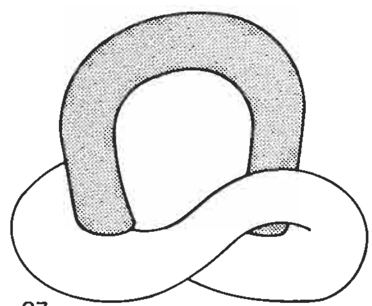

Fig. 37

Now that the twisting starts, the mirror symmetry is destroyed. Instead, there is a vertical axis of two-fold rotation symmetry; the surface is made up of two congruent halves, differing by a 180° rotation.

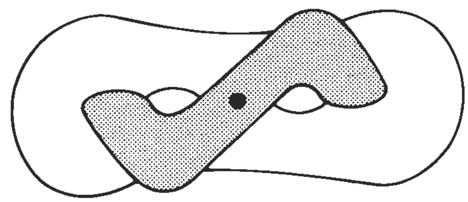

Fig. 38

Here is a top view, showing the twisted tube. The dot shows the position of the two-fold rotation axis. A rotation of 180°about this axis will take the set of points on the surface into the same set.

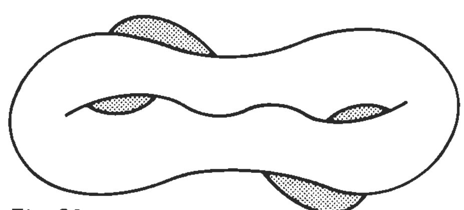

Fig. 39

Here is a view from the bottom, showing the two "ears" being twisted.

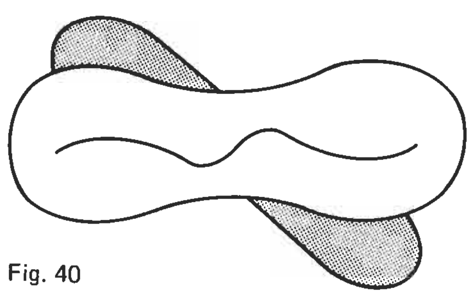

Fig. 40

Here the "ears" are twisted still more. You can no longer see the "eardrums." The bottom arc of the vertical self-intersection circle has become quite curved.

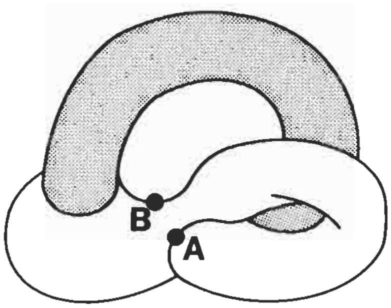

Fig. 41

Here is another side view. Notice that point A from the originally vertical intersection circle has moved close to point B which is from the originally horizontal intersection circle.

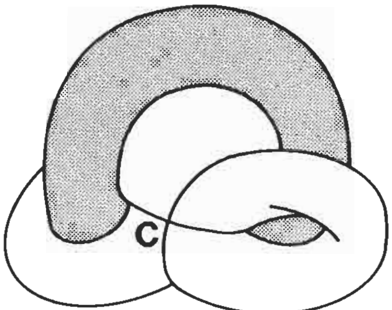

Fig. 42

Here is the next stage, an isthmus singularity as in Figure 16. The surfaces have become tangent at point C where A and B come together. A similar situation takes place on the other side of the model.

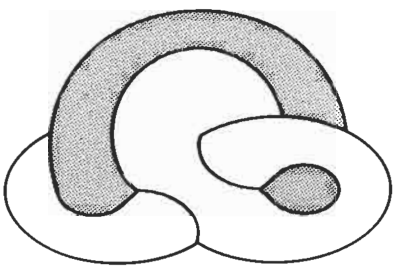

Fig. 43

If we go slightly past this singularity, the surface which previously separated points A and B is now partly hidden, and the other surface is now continuously visible.

The next stages are all topologically equivalent.

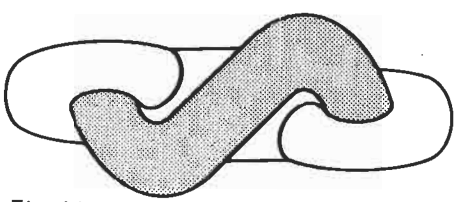

Fig. 44

In the top view we can see that the tube where the inside surface shows has been lengthened and further twisted.

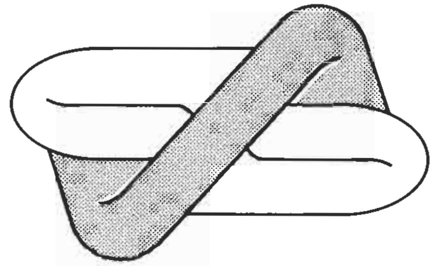

Fig. 45

Now the ears have been twisted more than 180°, and we can see a little into the holes from the top view, instead of from the bottom view. Also, the two earlobes line up to form a new tube, running under the old tube and back out again.

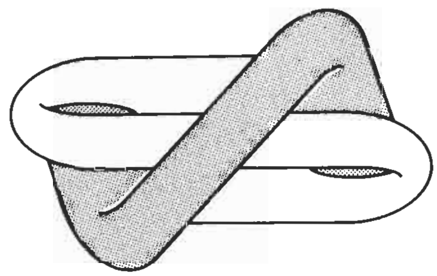

Fig. 46

Here is still further twisting. Small pieces of the eardrums are visible. Also the new tube now runs straight across, and even intersects, the old tube. We have passed a new singularity, which can be seen better from the side.

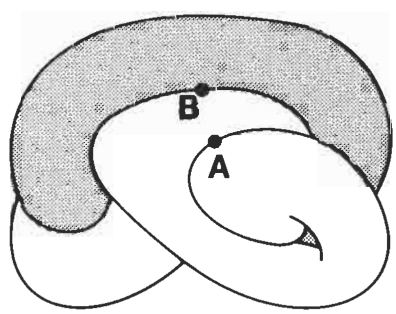

Fig. 47

Here is a side view again, slightly twisted from Figure 43. The top of the earlobe A continues into the surface and comes back out as the corresponding earlobe on the back side. Together they form another tube which has existed in a twisted form ever since the "isthmus" singularities were passed.

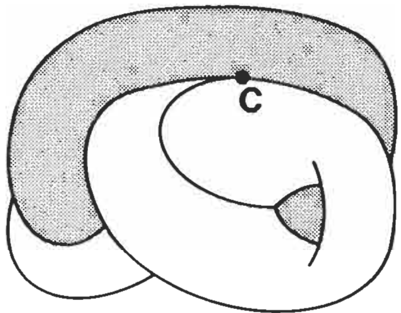

Fig. 48

This new tube rises and the old tube sinks, so that the points A and B meet in a new triple point C. The horizontal line running from left to right through C is tangent to all three surfaces at C; this is another singular triple point.

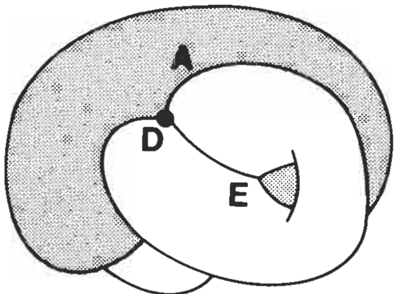

Fig. 49

As the new tube rises still further, C separates into two triple points. One is labeled D in the picture above. The other is hidden behind the earlobe. There are now four triple points which lie at the vertices of a curved tetrahedron. One edge of this tetrahedron is the arc DE.

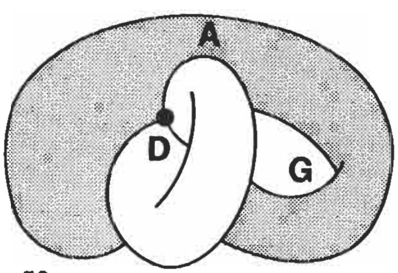

Fig. 50

As the tube containing A rises still further, this tetrahedron decreases in size and becomes proportionately less curved. In addition, further twisting takes place so that we can no longer see the eardrum containing E. However, we see a new eardrum G formed as the tube A cuts off a piece of the surface below B.

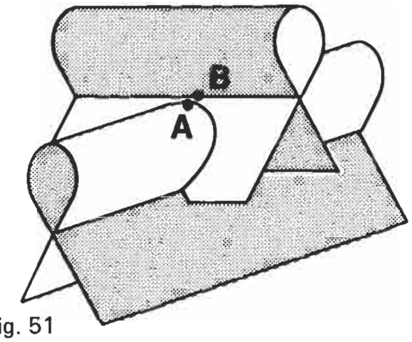

Fig. 51

The singularity may be easier to see if we just consider the pieces of tube near A and B and straighten them so that they are only curved in one direction. They could then be rolled from pieces of paper, if paper could pass through itself. Here is the stage just before the singularity.

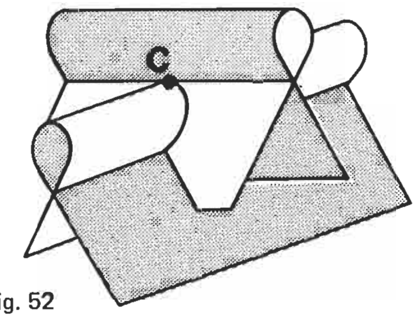

Fig. 52

Here the top of tube A has risen to become tangent to the straight line of intersection of the tube B. Thus all three surfaces are tangent to this horizontal line at the singular triple point C.

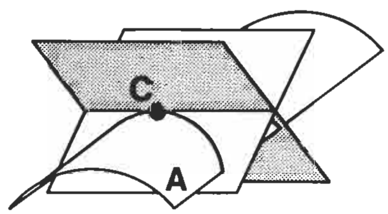

Fig. 53

Here is a close-up of the pieces of surface near C. This does not look much like the standard model in Figure 19 for such a triple point,

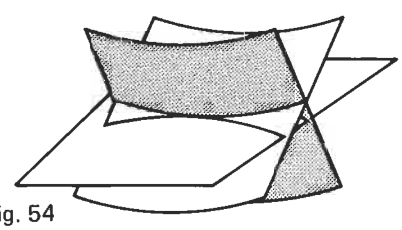

Fig. 54

but if we twist the picture by flattening the curved surface A and therefore curving the other two surfaces, we get a picture like the one above. This now looks more like an upside down version of the piece of our standard model near the triple point, in Figure 19.

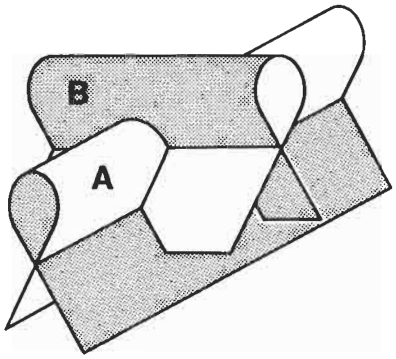

Fig. 55

If tube A rises still farther with respect to tube B, two new triple points are created. The four triple points form a tetrahedron.

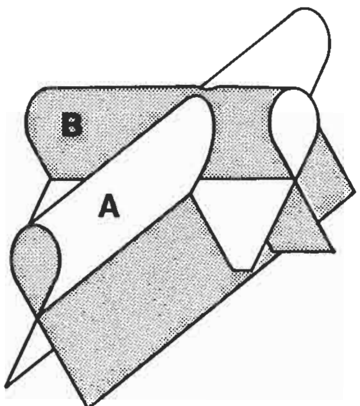

Fig. 56

As the tube A rises, this tetrahedron gets smaller. Also, as the top of tube A gets higher, more of it is visible. The place at the top of the tubes where their two intersection curves come together looks like the coastline of the isthmus. In fact, if we curl up the picture so the top of tube A flattens out and the top of tube B bends into a saddle shape, we can then get the standard model in Figure 14. This is similar to what we did with the triple point in Figure 53.

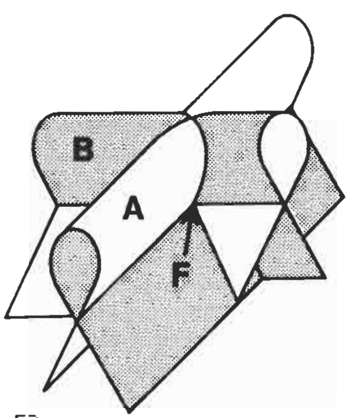

Fig. 57

When the two tubes are at the same level, they intersect as in the vault where two semicircular arches cross in Roman architecture. At the same time, the four triple points meet at a quadruple point F, where four planes intersect.

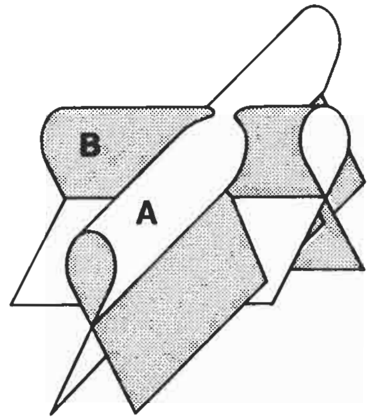

Fig. 58

When tube A (the sea) rises above tube B (the land), the intersection curves (the shoreline) near the top are joined in the opposite way. The quadruple point also opens again into a tetrahedron of the opposite orientation.

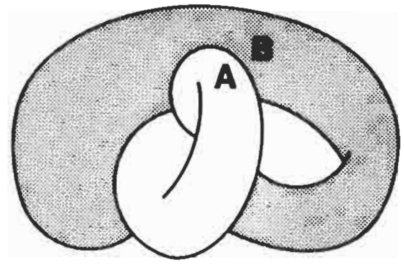

Fig. 59

Here is the same thing taking place on the whole sphere. The two tubes are now curved instead of straight, but the singularities take place the same way.

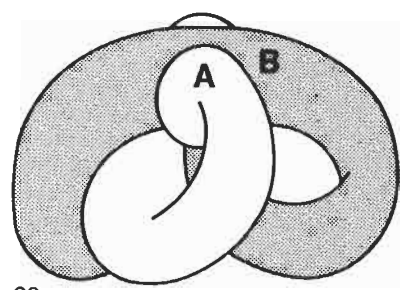

Fig. 60

As the two tubes approach each other, their top lines of intersection also approach until they join, forming the vaulted arch.

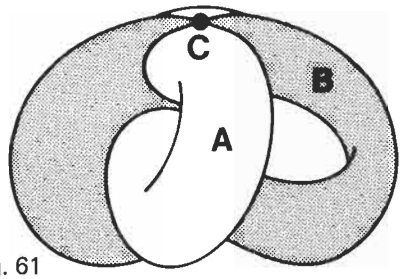

Fig. 61

The top point C is then a place where both surfaces have the same tangent plane. The model at this stage has four-fold symmetry, because a rotation of 90° about the vertical axis through C takes the model into the same shape.

However, this rotation reverses the inside and outside surfaces, because tube A (which came from the brim of the hat) was the original outside surface and tube B (which came from the top of the hat) had the inside surface visible. The sphere is now halfway inside out, because just as much of the inside surface is showing as the outside.

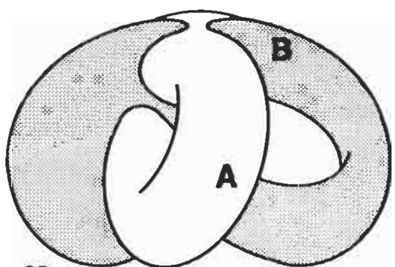

Fig. 62

When tube A rises still farther, it lies above tube B, which now intersects it in the same way tube A intersected tube B previously. If it continues to rise, we can see the rest of the process in reverse, with two of the triple points disappearing.

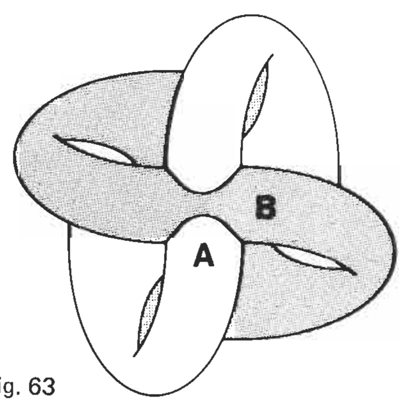

Fig. 63

The four-fold symmetry is easier to understand if the surfaces are viewed from above. Here is the surface just before the halfway stage, with the tube A below the tube B. You can see little pieces of the eardrums.

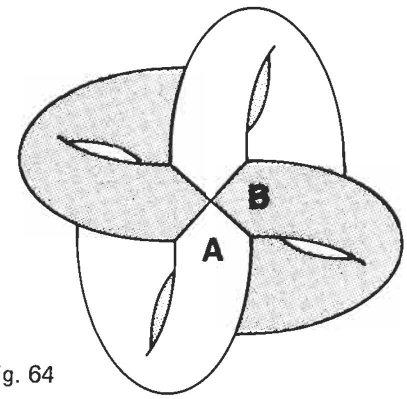

Fig. 64

Here is the halfway stage. A 90° rotation will interchange tubes A and B, and therefore the inside and outside surfaces. There are four equivalent ears.

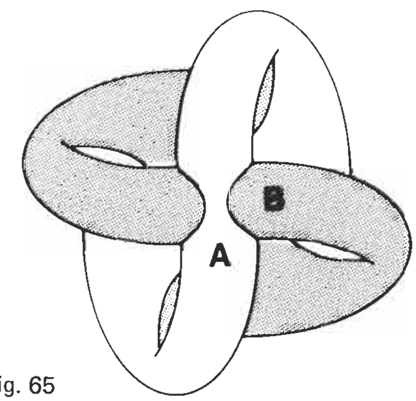

Fig. 65

Here is the surface just after the halfway stage. A 90° rotation of this picture produces the picture preceeding the halfway stage, with the outside tube A in the position of B, where the inside surface shows.

Thus the rest of the regular homotopy continues from the halfway stage in reverse, and the homotopy is symmetrical in time, by switching inside and outside. When all the steps are repeated, we will get a round, inside-out sphere.

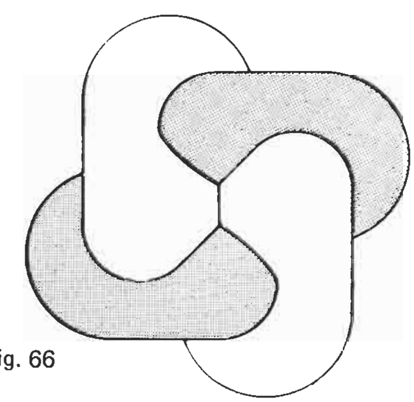
Fig. 66

Here are the same three stages, viewed from the bottom. When tube A is below tube B, a short segment of its intersection curve is visible from the bottom. It appears vertical in this view. This segment is the one visible edge of the tetrahedron formed by the four planes which slant inward toward the center. Its four vertices are the triple points where three of the planes meet.

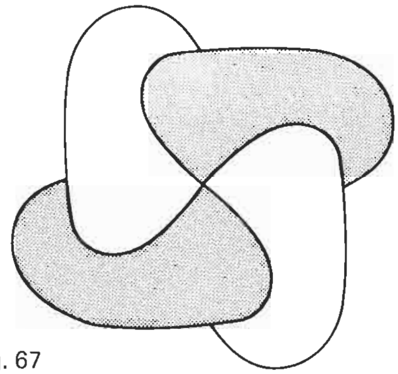

Fig. 67

When the two tubes are at the same level, the vertices of the tetrahedron come together in a single point, the quadruple point. This surface now has four-fold symmetry.

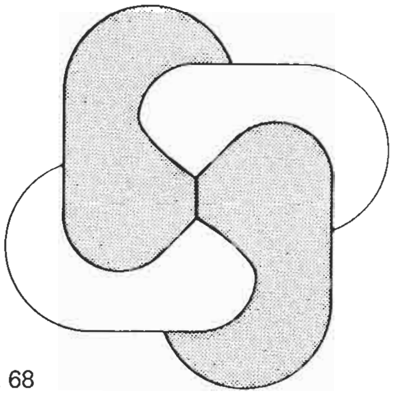

Fig. 68

When tube A is above tube B, B is closer to us as we view the model from below, and the small horizontal segment is a piece of its intersection curve. The quadruple point has split into four triple points again.

The motion then continues through the rest of the stages in reverse.

Altogether there are 14 singularities:

1) the creation of the first intersection circle,
2) the creation of the second intersection circle,
3) the creation of the first two triple points,
4&5) the simultaneous flooding of the two isthmuses,
6) the creation of the second two triple points
7&8) the simultaneous quadruple point and isthmus at the halfway stage,
9) the merging and disappearance of two triple points,
10&11) the exposing of two isthmuses,
12) the merging and disappearance of the last two triple points,
13) the disappearance of one circle of intersection, and
14) the disappearance of the last intersection circle.

No regular homotopy has been discovered with less than 14 singularities, so this one is among the simplest known.

References:

Smale's paper[1] first proved, using induction, that the sphere could be turned inside out in three dimensional space, and in fact, classified the positions of the ordinary sphere in a euclidean space of any dimension. The ideas in this paper were derived from an earlier one,[2] which studied regular homotopies of curves on an arbitrary surface, and were generalized in a later paper,[3] which considered spheres of arbitrary dimension. These three papers are rather technical, and the basic idea is explicated in the French mimeographed notes[4] of René Thom.

A more popular presentation[5] by Anthony Phillips gives a series of illustrations of a different regular homotopy.

[1] Stephen Smale, "A Classification of Immersions of the Two-sphere," *Transactions of the American Mathematical Society*, 90 (1959), 281.

[2] Stephen Smale, "Regular Curves on Riemannian Manifolds," *Transactions of the American Mathematical Society* 87 (1958), 492.

[3] Stephen Smale, "The Classification of Immersions of Spheres in Euclidean Space," *Annals of Mathematics* (2) 69 (1959), 327.

[4] René Thom, "La Classification des Immersions," Seminaire Bourbaki Exposé 157 (December 1957), Secrétariat Mathématique, Paris (1958).

[5] Anthony Phillips, "Turning a Surface Inside Out," *Scientific American* 214 (May 1966), 112.

The figures used in this guide were executed by Michael Hemby and William Urian.