Differential Geometry of Curves and Surfaces

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Preface

What is Differential Geometry?

Differential geometry studies the properties of curves, surfaces, and higher-dimensional curved spaces using tools from calculus and linear algebra. Just as the introduction of calculus expands the descriptive and predictive abilities of nearly every field of scientific study, the use of calculus in geometry brings about avenues of inquiry that extend far beyond classical geometry.

Before the advent of calculus, much of geometry consisted of proving consequences of Euclid's postulates. Even conics, which came into vogue in the physical sciences after Kepler observed that planets travel around the sun in ellipses, arise as the intersection of a double cone and a plane, two shapes that fit comfortably within the paradigm of Euclidean geometry. One cannot underestimate the impact of geometry on science, philosophy, and civilization as a whole. Not only did the geometry books in Euclid's *Elements* serve as the model for mathematical proof for over two thousand years in the Western tradition of a liberal arts education, but geometry also produced an unending flow of applications in surveying, architecture, ballistics, astronomy, astrology, and natural philosophy more generally.

The objects of study in Euclidean geometry (points, lines, planes, circles, spheres, cones, and conics) are limited in what they can describe. A boundless variety of curves and surfaces and manifolds arise naturally in areas of inquiry that employ geometry. To address these new classes of objects, various branches of mathematics brought their tools to bear on the expanding horizons of geometry, each with a different bent and set of fruitful results. Techniques from calculus and analysis led to differential geometry, pure set theoretic methods led to topology, and modern algebra contributed the field of algebraic geometry.

The types of questions one typically asks in differential geometry extend far beyond what one can ask in classical geometry and yet the former do not entirely subsume the latter. Differential geometry questions often fall into two categories: local properties, by which one means properties of a curve or surface defined in the neighborhood of a point, or global properties, which refer to properties of the curve or surface taken as a whole. As a comparison to functions of one variable, the derivative of a function f at a point a is a local property, since one only needs information about f near a, whereas the integral of f between a and b is a global property. Some of the most interesting theorems in differential geometry relate local properties to global ones. As a case in point, the celebrated Gauss-Bonnet Theorem single-handedly encapsulates many global results of curves in the plane at the same time as it proves results about spherical and hyperbolic geometry.

Using This Textbook

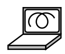
This book is the first in a pair of books that together are intended to bring the reader through classical differential geometry into the modern formulation of the differential geometry of manifolds. The second book in the pair, by Lovett, is entitled *Differential Geometry of Manifolds with Applications to Physics* [22]. Neither book directly relies on the other but knowledge of the content of this book is quite beneficial for [22].

On its own, the present book is intended as a textbook for a single semester undergraduate course in the differential geometry of curves and surfaces, with only vector calculus and linear algebra as prerequisites. The interactive computer graphics applets that are provided for this book can be used for computer labs, in-class illustrations, exploratory exercises, or simply as intuitive aides for the reader. Each section concludes with a collection of exercises that range from perfunctory to challenging and are suitable for daily or weekly problem sets.

However, the self-contained text, the careful introduction of concepts, the many exercises, and the interactive computer graphics also make this text well-suited for self-study. Such a reader should feel free to primarily follow the textbook and use the software as supporting material; to primarily follow the presentation in the software package and consult the textbook for definitions, theorems, and proofs; or to try to follow both with equal weight. Either way, the authors hope that the dual nature of the software applets and the classic textbook structure will offer the reader both a rigorous and intuitive introduction to the field of differential geometry.

Computer Applets

An integral part of this book is the access to online computer graphics applets that illustrate many concepts and theorems introduced in the text. Though one can explore the computer demos independently of the text, the two are intended as complementary modes of studying the same material: a visual/intuitive approach and an analytical/theoretical approach. Though the text does its best to explain the reason for various definitions and why one might be interested in such and such topic, the graphical applets can often provide motivation for certain definitions, allow the reader to explore examples further, and give a visual explanation for complicated theorems. The ability to change the choice of the parametric curve or the parametrized surface in an applet or to change other properties allows the reader to explore the concepts far beyond what a static book permits.



Any element in the text (Example, Exercise, Definition, Theorem, etc.) that has an associated applet is indicated by the symbol shown in this margin. Each demo comes with some explanatory text. The authors intended the applets to be intuitive enough so that after using just one or two (and reading the supporting text), any reader can quickly understand their functionality. However, the applets are extensible in that they are designed with considerable flexibility so that the reader can often change whether certain elements are displayed or not. Often, there are additional elements that one can display either by accessing the Controls menu on the Demo window or the Plot/Add Plot menu on any display window.

The authors encourage the reader to consult the tutorial page for the applets. All of the applet materials are available online at http://www.akpeters.com/DiffGeo.

Organization of Topics

Chapters 1 through 4 cover alternatively the local and global theory of plane and space curves. In the local theory, we introduce the fundamental notions of curvature and torsion, construct various associated objects (e.g., the evolute, osculating circle, osculating sphere), and present the fundamental theorem of plane or space curves, which is an analog of the fundamental theorem of calculus. The global theory studies how local properties (especially curvature) relate to global properties such as closedness, concavity, winding numbers, and knottedness. The topics in these chapters are particularly well suited for computer investigation. The authors know from experience in teaching how often students make discoveries on their own by being able to quickly manipulate curves and their associated objects and properties.

Chapter 5 rigorously introduces the notion of a regular surface, the type of surface on which the techniques of differential geometry are well defined. Here one first sees the tangent plane and the concept of orientability.

Chapter 6 introduces the local theory of surfaces in $\mathbb{R}^3$, focusing on the metric tensor and the Gauss map from which one defines the essential notions of principal, Gaussian curvature, and mean curvature. In addition, we introduce the study of surfaces that have Gaussian curvature or mean curvature identically 0. One cannot underestimate the importance of this chapter. Even a reader primarily interested in the advanced topic of differentiable manifolds should be comfortable with the local theory of surfaces in $\mathbb{R}^3$ because it provides many visual and tractable examples of what one generalizes in the theory of manifolds. Here again, as in Chapter 8, the use of the software applets is an invaluable aid for developing a good geometric intuition.

Chapter 7 first introduces the reader to the component notation for tensors. It then establishes the famous Theorema Egregium, the celebrated classical result that the Gaussian curvature depends only on the metric tensor. Finally, it outlines a proof for the fundamental theorem of surface theory.

Another title commonly used for Chapter 8 is *Intrinsic Geometry*. Just as Chapter 1 considers the local theory of plane curves, Chapter 8 starts with the local theory of curves on surfaces. Of particular importance in this chapter are geodesics and geodesic coordinates. The book culminates with the famous Gauss-Bonnet Theorem, both in its local and global forms, and presents applications to problems in spherical and hyperbolic geometry.

A Comment on Prerequisites

The mathematics or physics student often first encounters differential geometry at the graduate level. Furthermore, at that point, one is typically immediately exposed to the formalism of manifolds, thereby skipping the intuitive and visual foundation that informs the deeper theory. Indeed, the advent of computer graphics has added a new dimension to and renewed the interest in classical differential geometry. The authors wish to provide a book that introduces the undergraduate student to an interesting and visually stimulating mathematical subject that is accessible with only the typical calculus sequence and linear algebra as prerequisites.

In calculus courses, one typically does not study all the analysis that underlies the theorems one uses. Similarly, in keeping with the stated requirements, this textbook does not always provide all the topological and analytical background for some theorems. The reader who is interested in all the supporting material is encouraged to consult [22].

A few key results presented in this textbook rely on theorems from the theory of differential equations, but either the calculations are all spelled out or a reference to the appropriate theorem has been provided. Therefore, experience with differential equations is occasionally helpful though not at all necessary. In a few cases, the authors choose not to supply the full proofs of certain results but instead refer the reader to the more complete text [22].

A few exercises also require some skills beyond the stated prerequisites but these are clearly marked. We have marked exercises that require ordinary differential equations with (ODE). Problems marked with (*) indicate difficulty that may be related to technical ability, insight, or length.

Notation

A quick perusal of the literature on differential geometry shows that mathematicians and physicists usually present topics in this field in very different ways. In addition, the classical and modern formulations of many differential geometric concepts vary significantly. Whenever different notations or modes of presentation exist for a topic (e.g., differentials, metric tensor, tensor fields), this book attempts to provide an explicit coordination between the notation variances.

As a comment on vector notation, this book and [22] consistently use the following conventions. A vector or vector function in a Euclidean vector space is denoted by $\vec{v}$, $\vec{X}(t)$, or $\vec{X}(u,v)$. Often γ indicates a curve parametrized by $\vec{X}(t)$ while writing $\vec{X}(t) = \vec{X}(u(t), v(t))$ indicates a curve on a surface. The unit tangent and the binormal vectors of a curve in space are written in the standard notation $\vec{T}(t)$ and $\vec{B}(t)$, respectively, but the principal normal is written $\vec{P}(t)$, reserving $\vec{N}(t)$ to refer to the unit normal vector to a curve on a surface. For a plane curve, $\vec{U}(t)$ is the vector obtained by rotating $\vec{T}(t)$ by a positive quarter turn. Furthermore, we denote by $\kappa_g(t)$ the curvature of a plane curve as one identifies this curvature as the geodesic curvature in the theory of curves on surfaces.

In this book, we often work with matrices of functions. The functions themselves are denoted, for example, by a_{ij}, and we denote the matrix by (a_{ij}). Furthermore, it is essential to distinguish between a linear transformation between vector spaces $T : V \to W$ and its matrix with respect to given bases in V and W. Following notation that is common in current linear algebra texts, if $\mathcal{B}$ is a basis in V and $\mathcal{B}'$ is a basis in W, then we denote by $[T]_{\mathcal{B}}^{\mathcal{B}'}$ the matrix of T with respect to these bases. If the bases are understood by context, we simply write $[T]$ for the matrix associated to T.

Occasionally, there arise irreconcilable discrepancies in definitions or notations (e.g., the definition of a critical point for a function $\mathbb{R}^n \to \mathbb{R}^m$, how one defines θ and ϕ in spherical coordinates, what units to use in electromagnetism). In these instances the authors made a choice that best suits their purposes and indicated commonly used alternatives.