Differential Geometry of Curves and Surfaces

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CHAPTER 1

Plane Curves: Local Properties

Just as one studies real functions of one variable before tackling multivariable calculus, so it makes sense to study curves before studying surfaces and higher-dimensional objects. This first chapter presents local properties of plane curves, where by a *local property* one means properties that are defined in a neighborhood of a point on the curve. For the sake of comparison with calculus, the derivative $f'(a)$ of a function f at a point a is a local property of the function since one only needs knowledge of $f(x)$ for $x \in (a-\varepsilon, a+\varepsilon)$ to define $f'(a)$. In contrast, the definite integral of a function over an interval is a global property since one needs knowledge of the function over the whole stated interval to calculate the integral. In contrast to this present chapter, Chapter 2 introduces global properties of plane curves.

1.1 Parametrizations

Borrowing from a physical understanding of motion in the plane, one could think about plane curves by specifying at time t the coordinates x and y of the position of a point traveling along the curve. Thus we need two functions $x(t)$ and $y(t)$. Using vector notation to locate a point on the curve, we often write $\vec{x}(t) = (x(t), y(t))$ for the above pair and call $\vec{x}(t)$ a *vector function* into $\mathbb{R}^2$. From a mathematical standpoint, t does not have to refer to time and is simply called the *parameter* of the vector function.

Example 1.1.1 (Lines). In basic analytic geometry one learns that every line in the plane can be uniquely specified by two nonequal points $\vec{p}_1 = (x_1, y_1)$ and $\vec{p}_2 = (x_2, y_2)$. The vector

$$\vec{v} = \vec{p}_2 - \vec{p}_1 = (x_2 - x_1, y_2 - y_1)$$

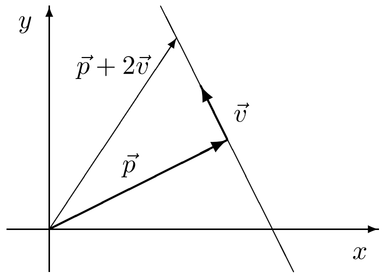

Figure 1.1. A line in the plane.

is called a *direction vector* of the line because it points along the same orientation in the plane as the line does. Any other direction vector of the line is a nonzero multiple of $\vec{v}$. Then every point on the line can be written with a position vector $\vec{p}_1 + t\vec{v}$ for some $t \in \mathbb{R}$. Therefore, we find that a line can also be defined by providing a point and a direction vector.

Using the coordinates of vectors, given a point $\vec{p} = (x_0, y_0)$ and a direction vector $\vec{v} = (v_1, v_2)$, a line through $\vec{p}$ in the direction of $\vec{v}$ is the image of the following vector function:

$$\vec{x}(t) = \vec{p} + t\vec{v} = (x_0 + v_1 t, y_0 + v_2 t).$$

Example 1.1.2 (Circles). The pair of functions

$$\vec{x}(t) = (R\cos t + a, R\sin t + b)$$

trace out a circle of radius R about the point (a, b). To see this, note that by the definition of the $\sin t$ and $\cos t$ functions, $(\cos t, \sin t)$ are the coordinates of the point on the unit circle that is also on the ray out of the origin that makes an angle t with the positive x-axis. Thus,

$$\vec{x}_1(t) = (\cos t, \sin t), \qquad \text{with } t \in [0, 2\pi],$$

traces out the unit circle. Multiplying both coordinate functions by R stretches the circle out by a factor of R away from the origin. Thus, the vector function

$$\vec{x}_2(t) = (R\cos t, R\sin t), \qquad \text{with } t \in [0, 2\pi],$$

has as its image the circle of radius R centered at the origin. Notice also that by writing $\vec{x}_2(t) = (x(t), y(t))$, we deduce that $x(t)^2 + y(t)^2 = R^2$ for all t, which is the algebraic equation of the circle. In order to obtain a vector function that traces out a circle centered at the point (a, b), we must simply translate $\vec{x}_2$ by the vector (a, b). This is simply vector addition, and so we get

$$\vec{x}(t) = (R\cos t + a, R\sin t + b), \qquad \text{with } t \in [0, 2\pi].$$

Two different vector functions can have the same image in $\mathbb{R}^2$. For example,

$$\vec{x}(t) = (\cos \omega t, \sin \omega t), \qquad \text{with } t \in [0, 2\pi],$$

also has the unit circle as its image. However, referring to vocabulary in physics, this latter vector function corresponds to a point moving around the unit circle at an angular velocity of ω.

When trying to establish a suitable mathematical definition of what one usually thinks of as a curve, one does not wish to consider as a curve points that jump around or pieces of segments. We would like to think of a curve as unbroken in some sense. In calculus, one introduces the notion of continuity to describe functions without "jumps" or holes, but one must exercise a little care in carrying over the notion of continuity to vector functions. More generally, we need to define the notion of a limit of a vector function as the parameter t approaches a fixed value. First, however, we remind the reader of the Euclidean distance formula.

Definition 1.1.3. Let $\vec{v}$ be a vector in $\mathbb{R}^n$ with coordinates $\vec{v} = (v_1, v_2, \dots, v_n)$ in the standard basis. The (Euclidean) length of $\vec{v}$ is given by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

If p and q are two points in $\mathbb{R}^n$ with coordinates given by vectors $\vec{v}$ and $\vec{w}$, then the Euclidean distance between p and q is $\|\vec{w} - \vec{v}\|$.

Definition 1.1.4. Let $\vec{x}$ be a vector function from a subset of $\mathbb{R}$ into $\mathbb{R}^n$. We say that the limit of $\vec{x}(t)$ as t approaches a is a vector $\vec{w}$, and we write

$$\lim_{t \to a} \vec{x}(t) = \vec{w},$$

if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |t-a| < \delta$ implies $\|\vec{x}(t) - \vec{w}\| < \varepsilon$.

Definition 1.1.5. Let I be an open interval of $\mathbb{R}$, let $a \in I$, and let $\vec{x} : I \to \mathbb{R}^2$ be a vector function. We say that $\vec{x}(t)$ is continuous at a if the limit as t approaches a of $\vec{x}(t)$ exists and

$$\lim_{t\to a} \vec{x}(t) = \vec{x}(a).$$

The above definitions mirror the usual definition of a limit of a real function but must use the length of a vector difference to discuss the proximity between $\vec{x}(t)$ and a fixed vector $\vec{w}$. Though at the outset this definition appears more complicated than the usual definition of a limit of a real function, the following proposition shows that it is not.

Proposition 1.1.6. *Let $\vec{x}$ be a vector function from a subset of $\mathbb{R}$ into $\mathbb{R}^n$ that is defined over an interval containing a, though perhaps not at a itself. Suppose in coordinates we have $\vec{x}(t) = (x(t), y(t))$ wherever $\vec{x}$ is defined. If $\vec{w} = (w_1, w_2)$, then $\lim_{t\to a} \vec{x}(t) = \vec{w}$ if and only if $\lim_{t\to a} x(t) = w_1$ and $\lim_{t\to a} y(t) = w_2$.*

Proof: Suppose first that $\lim_{t\to a} \vec{x}(t) = \vec{w}$. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ satisfy the definition of the limit of the vector function. Note that $|x(t) - w_1| < \|\vec{x}(t) - \vec{w}\|$ and that $|y(t) - w_2| < \|\vec{x}(t) - \vec{w}\|$. Hence, $0 < |t-a| < \delta$ implies $|x(t) - w_1| < \varepsilon$ and $|y(t) - w_2| < \varepsilon$. Thus, $\lim_{t\to a} x(t) = w_1$ and $\lim_{t\to a} y(t) = w_2$.

Conversely, suppose that $\lim_{t\to a} x(t) = w_1$ and $\lim_{t\to a} y(t) = w_2$. Let $\varepsilon > 0$ be an arbitrary positive real number. By definition, there exist δ_1 and δ_2 such that $0 < |t-a| < \delta_1$ implies $|x(t) - w_1| < \varepsilon/\sqrt{2}$ and $0 < |t-a| < \delta_2$ implies $|y(t) - w_2| < \varepsilon/\sqrt{2}$. Taking $\delta = \min(\delta_1, \delta_2)$ we see that $0 < |t-a| < \delta$ implies that

$$\|\vec{x}(t) - \vec{w}\| = \sqrt{|x(t) - w_1|^2 + |y(t) - w_2|^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon.$$

This finishes the proof of the proposition. □

Corollary 1.1.7. *Let I be an open interval of $\mathbb{R}$, let $a \in I$, and consider a vector function $\vec{x} : I \to \mathbb{R}^2$ with $\vec{x}(t) = (x(t), y(t))$. Then $\vec{x}(t)$ is continuous at $t = a$ if and only if $x(t)$ and $y(t)$ are continuous at $t = a$.*

Definition 1.1.5 and Corollary 1.1.7 provide the mathematical framework for what one usually thinks of as a curve in physical intuition. This motivates the following definition.

Definition 1.1.8. Let I be an interval of $\mathbb{R}$. A *parametrized curve* (or *parametric curve*) in the plane is a continuous function $\vec{x} : I \to \mathbb{R}^2$. If we write $\vec{x}(t) = (x(t), y(t))$, then the functions $x : I \to \mathbb{R}$ and $y : I \to \mathbb{R}$ are called the *coordinate functions* or *parametric equations* of the parametrized curve. We call the *locus* of $\vec{x}(t)$ the image of $\vec{x}(t)$ as a subset of $\mathbb{R}^2$.

The following examples begin to provide a library of parametric curves and illustrate how to construct parametric curves to describe a particular shape or trajectory.

Example 1.1.9 (Graphs of Functions). The graph of a continuous function $f : [a, b] \to \mathbb{R}$ over an interval $[a, b]$ is a parametric curve. In order to view the graph of a continuous function as a parametrized curve, we use the coordinate functions $\vec{x}(t) = (t, f(t))$, with $t \in [a, b]$.

Example 1.1.10 (Circles Revisited). Another parametrization for the unit circle (sometimes used in number theory) is

$$\vec{x} = (x(t), y(t)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \qquad \text{for } t \in \mathbb{R}. \tag{1.1}$$

It is easy to see that for all $t \in \mathbb{R}$, $x(t)^2 + y(t)^2 = 1$, which means that the locus of $\vec{x}$ is on the unit circle. However, this parametrization does not trace out the entire circle as it misses the point $(-1, 0)$. We leave it as an exercise to determine a geometric interpretation of the parameter t and to show that

$$\lim_{t \to \infty} \vec{x} = \lim_{t \to -\infty} \vec{x} = (-1, 0).$$

Example 1.1.11 (Ellipses). Without repeating all the reasoning of the previous exercise, it is not hard to see that

$$\vec{x}(t) = (a \cos t, b \sin t)$$

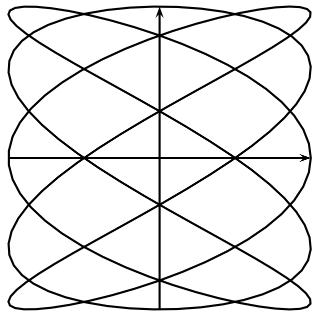

Figure 1.2. A Lissajous figure.

provides a parametrization for the ellipse centered at the origin with axes along the x- and y-axes, with respective half-axes of length a and b. Note that these coordinate functions do indeed satisfy

$$\frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} = 1 \qquad \text{for all } t.$$

Example 1.1.12 (Lissajous Figures). It is sometimes amusing to see how $\cos t$ and $\sin t$ relate to each other if we change their respective periods. Lissajous figures, which arise in the context of electronics, are curves parametrized by

$$\vec{x}(t) = (\cos mt, \sin nt),$$

where m and n are positive integers. See Figure 1.2 for an example of a Lissajous figure with $m = 5$ and $n = 3$.

Example 1.1.13 (Cycloids). We can think of a regular cycloid as the locus traced out by a point of light affixed to a bicycle tire as the bicycle rolls forward. We can establish a parametrization of such a curve as follows.

Assume the wheel of radius a begins with its center at $(0, a)$ so that the part of the wheel touching the x-axis is at the origin. We view the wheel as rolling forward on the positive x-axis. As the wheel rolls, the position of the center of the wheel is $\vec{f}(t) = (at, a)$, where t is the angle measuring how much (many times) the wheel has turned since it started. At the same time, the light—at a distance a from

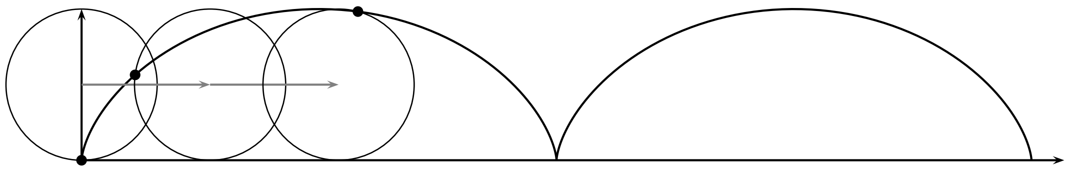

Figure 1.3. Cycloid.

the center of the wheel and first positioned straight down from the center of the wheel—rotates in a clockwise motion around the center of the wheel. The motion of the light with respect to the center of the wheel is

$$\vec{g}(t) = \left(a\cos\left(-t - \frac{\pi}{2}\right), a\sin\left(-t - \frac{\pi}{2}\right)\right) = (-a\sin t, -a\cos t).$$

The locus of the cycloid is the vector function that is the sum of $\vec{f}(t)$ and $\vec{g}(t)$. Thus, a parametrization for the cycloid is

$$\vec{x}(t) = (at - a\sin t, a - a\cos t).$$

One can point out that reflectors on bicycle wheels are usually not attached directly on the tire but on a spoke of the wheel. We can easily modify the above discussion to obtain the relevant parametric equations for when the point of light is located at a distance b from the center of the rolling wheel. One obtains

$$\vec{x}(t) = (at - b\sin t, a - b\cos t).$$

If $0 < b < a$, one obtains the curve of a realistic bicycle tire reflector, and this locus is called a *curtate cycloid*. In contrast, the locus obtained by letting $b > a$ is called a *prolate cycloid*.

Example 1.1.14 (Heart Curve). Arguably the most popular curve around Valentine's Day is the heart. Here are some parametric equations that trace out such a curve:

$$\vec{x}(t) = ((1 - \cos^2 t)\sin t, (1 - \cos^3 t)\cos t).$$

We encourage the reader to visit this example in the accompanying software and to explore ways of modifying these equations to suit their purposes.