

Introduction to Part II

The first half of this book provided the details needed to classify the symmetries of repeating patterns on the plane and sphere using the method of orbifolds and the signature notation for them. Traditionally, this enumeration was done using group theory. In the central section of the book, we show how our method relates to the traditional one, by way of a discussion of color symmetries. We continue to provide examples and illustrations to accompany the material.

So far we've avoided group theory, except for an aside in Chapter 4 but we expect readers of Part II to know some group theory. We expect such readers will be surprised that we classified these groups without using any group theory! Surprisingly, group theory doesn't really help.

In the upcoming chapters we once again place the results before the theorems, this time discussing color symmetries using the ideas supported by the discussion in Chapter 14.

The new method we used for classifying repeating patterns was pioneered by Murray McBeath and is now recognized as part of the wide-ranging theory of groups and manifolds that William Thurston has made his own. Our signature is a shortened form of McBeath's, which conveys exactly the same information.

We have been following what one can call Thurston's commandment:

Thou shalt know no geometrical group save by understanding its orbifold.

Our orbifold signature is only one—which we hope is the last one—of several systems of names for these groups. Dictionaries between it and the other systems are given in Tables A.1 and A.2 of

the appendix. There is an important way in which it differs from all the other systems except McBeath's; namely, these systems usually name groups by somehow specifying their generators. After mathematicians have been thinking for a century, we can see that this was not a good way to proceed—after all, it disobeys Thurston's commandment! Because groups have many generating sets, each author has made a more or less arbitrary choice of which generators to use and how to indicate them. Moreover, since their systems are usually not theoretically complete, difficult cases must often be distinguished by adding arbitrary signs.

It is important to realize that the characters in the signature do *not* correspond to generators in this way. What they do name (in compliance with the commandment!) is features of the orbifold, which may correspond to local singularities (gyration points and kaleidoscopes) or global properties of the topology of the orbifold (wonders and miracles).

There is a vague relationship between these notions and the group elements that are responsible for them, but it's far from being a one-to-one correspondence and is usually misleading. For instance, there are two **2**'s in our name **22** for a certain group despite the fact that that group has just one order-2 rotation. Why? Because that rotation creates two singularities on the orbifold. Again, the two *****'s in ****** don't really correspond to a generating set. Each ***** is caused by a reflection, but those two reflections do not suffice to generate the group!

However, there must be a way to recover group generators from the signature since everything about the group can be read from its orbifold. We provide it in the next chapter, which sets forth the rules for obtaining group presentations from signatures. The following chapters then apply these presentations to the problem of finding symmetric "colorings" of repeating patterns.

The complete lists of all primefold colorings of repeating patterns in the plane and sphere appear for the first time in this book. You can understand these lists without needing to follow the technical arguments of Chapters 11–13 that deduce them from the group presentations. Following our practice, group presentations are finally justified in Chapter 14.

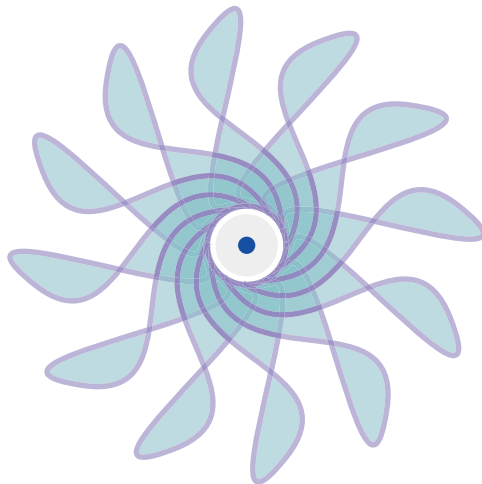
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We turn to two other subjects in the last two chapters of Part II. Chapter 15 uses the orbifold idea to enumerate interesting tilings, while Chapter 16, as a digression, enumerates the abstract groups.

Some Group Theory

In this next portion of the book we will be discussing the groups of symmetries of our patterns. Here we give a gentle introduction to mathematical group theory.

In Chapter 4 we said that the symmetries of a pattern form a group. This means something quite specific. Consider this gyroscopic pattern.



It has exactly twelve symmetries: we may rotate by 30° , 60° , ... on up to 330° . That makes eleven, and the twelfth symmetry—the *identity*—is the one that does nothing at all or, equivalently, rotates through 360° .

These symmetries may be combined: we may first rotate by, say, 240° and then by 150° ; the end result would be the same as rotating

by 390° or, more simply, 30° , which (of course) is also one of our twelve symmetries.

The identity is special: when we combine the identity with any other symmetry, we don't change the result. Also, every symmetry has an *inverse* that is its undoing; combining a symmetry with its inverse produces the identity.

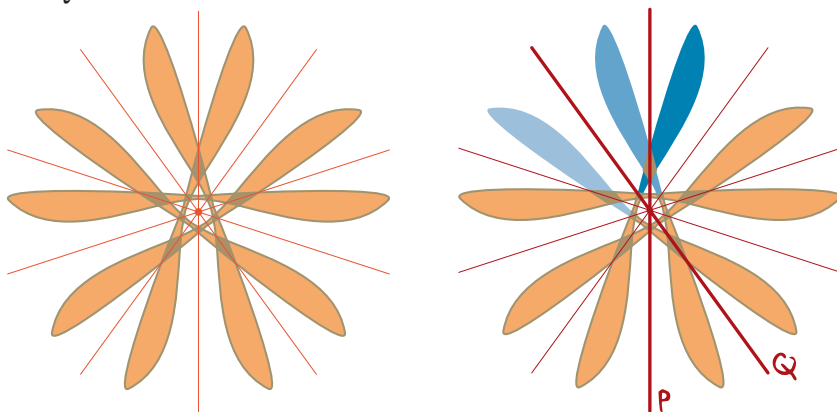
In general, a pair of symmetries A and B of a pattern have a product AB , obtained by performing the motions of the pattern corresponding to A and B one after the other, producing another symmetry in the group. This production is *associative*: performing motion AB followed by motion C is the same as performing motion A followed by motion BC . There is an identity symmetry, 1 , which doesn't move the pattern at all; in particular, for any symmetry A , $A1 = A$ and $1A = A$. Finally, for every symmetry A there is an inverse symmetry A^{-1} with $AA^{-1} = 1$ and $A^{-1}A = 1$. These conditions are precisely those that define a mathematical *group*. How much information is needed to specify a group? In the example above, all the symmetries could be formed by applying just one symmetry repeatedly—say, rotating by 30° —over and over again. We say that this symmetry, which we'll call α , *generates* the group.

Moreover, some symmetries can be made in many ways; for example, the rotation by 60° can be achieved by applying the rotation α twice, fourteen times, or twenty-six times. But knowing that $1 = \alpha^{12}$ suffices to explain this, and we call this equation a *relation* of the group.

A *group presentation* consists of a list of generators and relations that suffice to describe the group. The figure's group of gyroscopic symmetries is fully described by knowing it has one generator α and one relation $1 = \alpha^{12}$. We can summarize the presentation by writing $\langle \alpha \mid 1 = \alpha^{12} \rangle$.

In this next example, of a pattern with kaleidoscopic symmetry, there are ten symmetries: five are reflections across the marked mirror lines, four are rotations of 72° , 144° , 216° , and 288° , and the tenth is the identity. When we combine these, do we really get a symmetry on our list? If we reflect across line P and then Q , the deep blue lobe is first taken to the medium blue one and then on to the light blue one. The net effect is that we rotated counterclockwise by 72° .

Try combining other symmetries yourself! In general, applying first one and then another of the symmetries of any pattern will result in a symmetry. But, the order in which these symmetries are applied can make a difference! PQ amounts to a counterclockwise rotation but QP is clockwise.



This group can be presented with just two generators, P and Q —all the symmetries are formed from these—and three relations, $1 = P^2$, $1 = Q^2$, and $1 = (PQ)^5$, that explain all other equivalences. We can summarize this presentation by writing $\langle P, Q \mid 1 = P^2 = Q^2 = (PQ)^5 \rangle$.