Preface

It should be noted that children's games are not merely games. One should regard them as their most serious activities.

Michel Eyquem de Montaigne

Herein we study games of pure strategy, in which there are only two players[1] who alternate moves, without using dice, cards or other random devices and where the players have perfect information about the current state of the game. Familiar games of this type include: TIC TAC TOE, DOTS & BOXES, CHECKERS and CHESS. Obviously, card games such as GIN RUMMY, and dice games such as BACKGAMMON are not of this type. The game of BATTLESHIP has alternate play, and no chance elements, but fails to include perfect information — in fact that's rather the point of BATTLESHIP. The games we study have been dubbed *combinatorial games* to distinguish them from the games usually found under the heading of *game theory,* which are games that arise in economics and biology.

For most of history, the mathematical study of games consisted largely of separate analyses of extremely simple games. This was true up until the 1930s when the Sprague-Grundy theory provided the beginnings of a mathematical foundation for a more general study of games. In the 1970s, the twin tomes *On Numbers and Games* by Conway and *Winning Ways* by Berlekamp, Conway, and Guy established and publicized a complete and deep theory, which can be deployed to analyze countless games. One cornerstone of the theory is the notion of a disjunctive sum of games, introduced by John Conway for normal-play games. This scheme is particularly useful for games that split naturally into components. *On Numbers and Games* describes these mathematical ideas at a sophisticated level. *Winning Ways* develops these ideas, and many more, through playing games with the aid of many a pun and witticism. Both books

[1] In 1972, Conway's first words to one of the authors, who was an undergraduate at the time, was "What's $1 + 1 + 1$?" alluding to three-player games. This question has still not been satisfactorily answered.

have a tremendous number of ideas and we acknowledge our debt to the books and to the authors for their kind words and teachings throughout our careers.

The aim of our book is less grand in scale than either of the two tomes. We aim to provide a guide to the evaluation scheme for normal-play, two-player, finite games. The guide has two threads, the theory and the applications.

The theory is accessible to any student who has a smattering of general algebra and discrete math. Generally, a third year college student, but any good high school student should be able to follow the development with a little help. We have attempted to be as complete as possible, though some proofs in Chapters 8 and 9 have been omitted, because the theory is more complex or is still in the process of being developed. Indeed, in the last few months of writing, Conway prevailed on us to change some notation for a class of all-small games. This *uptimal* notation turned out to be very useful and it makes its debut in this book.

We have liberally laced the theory with examples of actual games, exercises and problems. One way to understand a game is to have someone explain it to you; a better way is to muse while pushing some pieces around; and the best way is to play it against an opponent. Completely solving a game is generally hard, so we often present solutions to only some of the positions that occur within a game. The authors invented more games than they solved during the writing of this book. While many found their way into the book, most of these games never made it to the rulesets found at the end. A challenge for you, the reader of our missive, and as a test of your understanding, is to create and solve your own games as you progress through the chapters.

Since the first appearance of *On Numbers and Games* and *Winning Ways* there have been several conferences specifically on combinatorial games. The subject has moved forward and we present some of these developments. However, the interested reader will need to read further afield to find the theories of loopy games, misère-play games, other (non-disjunctive) sums of games, and the computer science approach to games. The proceedings of these conferences [Guy91, Now96, Now02, FN04] would be good places to start.

Organization of the Book

The main idea of this part of the theory of combinatorial games is the assigning of values to games, values that can be used to replace the actual games when deciding who wins and what the winning strategies might be.

Each chapter has a prelude which includes problems for the student to use as a warm-up for the mathematics to be found in the following chapter. The prelude also contains guidance to the instructor for how one can wisely deviate from the material covered in the chapter.

Exercises are sprinkled throughout each chapter. These are intended to reinforce, and check the understanding of, the preceding material. Ideally then, a student should try every exercise as it is encountered. However, there should be no shame associated with consulting the solutions to the exercises found at the back of the book if one or more of them should prove to be intractable. If that still fails to clear matters up satisfactorily, then it may be time to consult a *games guru.*

Chapter 0 introduces basic definitions and loosely defines that portion of game theory which we will address in the book. Chapter 1 covers some general strategies for playing or analyzing games and is recommended for those who have not played many games. Others can safely skim the chapter and review sections on an as-needed basis while reading the body of the work. Chapters 2, 4, and 5 contain the core of the general mathematical theory. Chapter 2 introduces the first main goal of the theory, that being to determine a game's *outcome class* or who should win from any position. Curiously, a great deal of the structure of some games can be understood solely by looking at outcome classes. Chapter 3 motivates the direction the theory takes next. Chapters 4, 5, and 6 then develop this theory (i.e., assigning values and the consequences of these values.)

Chapters 7, 8, and 9 look at specific parts of the universe of combinatorial games and as a result, these are a little more challenging but also more concrete since they are tied more closely to actual games. Chapter 7 takes an in-depth look at *impartial* games. The study of these games pre-dates the full theory. We place them in the new context and show some of the new classes of games under present study. Chapter 8 addresses hot games, games such as GO and AMAZONS in which there is a great incentive to move first. There is much research in this area and we can only give an introduction to this material. Chapter 9 looks at the analysis of *all-small* games. Most of the research emphasis has been on impartial and hot games. Only recently have there been developments in this area and we present the original and latest results in light of all the new developments in combinatorial game theory.

Chapter $\boldsymbol{\omega}$ is a brief listing of other areas of active research that we could not fit into an introductory text.

In Appendix A, we present top-down induction, an approach that we use often in the text. While the student need not read the appendix in its entirety, the first few sections will help ground the format and foundation of the inductive proofs found in the text.

Appendix B is a brief introduction to CGSuite, a powerful programming toolkit written by Aaron Siegel in Java for performing algebraic manipulations on games. CGSuite is to the combinatorial game theorist what Maple or Mathematica is to a mathematician or physicist. While the reader need not use CGSuite while working through the text, the program does help to build intuition, double-check work done by hand, develop hypotheses, and handle some of the drudgery of rote calculations.

Appendix D contains the rules to any game in the text that either appears multiple times or is found in the literature. We do not always state the rules to a game within the text, so the reader will want to refer to this appendix often.

The supporting website for the book is located at www.lessonsinplay.com. Look there for links, programs, and addenda, as well as instructions for accessing the online solutions manual for instructors.

Acknowledgments

While we are listed as the *authors* of this text, we do not claim to be the main contributors. The textbook emerged from a mathematically rich environment created by others. We got to choose the words and consequently, despite the best efforts of friends and colleagues, all the errors are ours.

Many of the contributors to this environment are cited within the book. There were many that also contributed to and improved the contents of the text itself and who deserve special thanks. We are especially grateful to Elwyn Berlekamp, John Conway, and Richard Guy who encouraged — and, at times, hounded — us to complete the text, and we hope it helps spawn a new generation of active aficionados.

Naturally, much of the core material and development is a reframing of material in *Winning Ways* and *On Number and Games.* We have adopted some of the proofs of J. P. Grossman, particularly that of the *Number-Avoidance Theorem*. Aviezri Fraenkel contributed the *Fundamental Theorem of Combinatorial Games,* which makes its appearance at the start of Chapter 2. Dean Hickerson helped us to prove Theorem 6.19 on page 126, that a game with negative incentives must be a number. Conway repeatedly encouraged us to adopt the *uptimal* notation in Chapter 9, and it took us some time to see the wisdom of his suggestions. Elwyn Berlekamp and David Molnar contributed some fine problems. Paul Ottaway, Angela Siegel, Meghan Allen, Fraser Stewart, and Neil McKay were students who pretested portions of the book and provided useful feedback, corrections, and clarifications. Elwyn Berlekamp, Richard Guy, Aviezri Fraenkel, and Aaron Siegel edited various chapters of our work for technical content, while Christine Aikenhead edited for language. Brett Stevens and Chris Lewis read and commented on parts of the book. Susan Hirshberg contributed the title of our book.

In this age of large international publishers, A K Peters is a fantastic and refreshing publishing house to work with. While they appreciate and under-

stand the business of publishing, we are convinced they care more about the dissemination of fine works than about the bottom line.

We thank our spice[2] for their loving support, and Lila and Tovia, who are the real *Lessons in Play*.

[2]affectionate plural of spouse