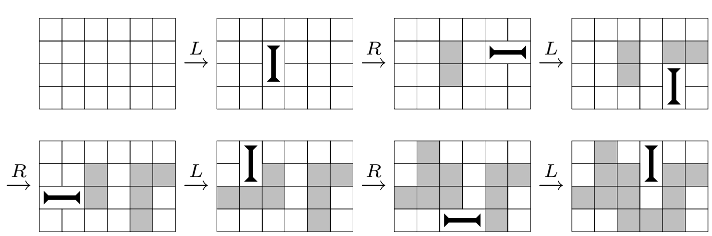
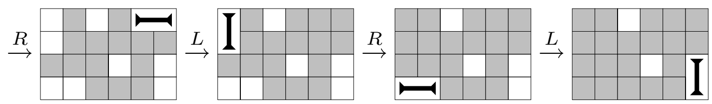
Chapter 0
Combinatorial Games

We don't stop playing because we grow old;
we grow old because we stop playing.

George Bernard Shaw

This book is all about *combinatorial games* and the mathematical techniques that can be used to analyze them. One of the reasons for thinking about games is so that you can be more skillful and have more fun playing them; so let's begin with an example called DOMINEERING. To play you will need a chessboard and a set of dominoes. The domino pieces should be big enough to cover or partially cover two squares of the chessboard but no more. You can make do with a chessboard and some slips of paper of the right size or even play with pen or pencil on graph paper (but the problem there is that it will be hard to undo moves when you make a mistake!). The rules of DOMINEERING are simple. Two players alternately place dominoes on the chessboard. A domino can only be placed so that it covers two adjacent squares. One player, Louise, places her dominoes so that they cover vertically adjacent squares. The other player, Richard, places his dominoes so that they cover horizontally adjacent squares. The game ends when one of the players is unable to place a domino, and that player then loses. Here is a sample game on a 4×6 board with Louise moving first:

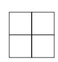
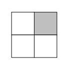




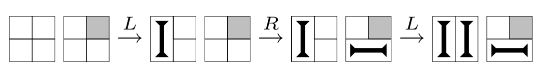
Since Louise placed the last domino, she has won.

Exercise 0.1. Stop reading! Find a friend and play some games of DOMINEERING. A game on a full chessboard can last a while so you might want to play on a 6×6 square to start with.

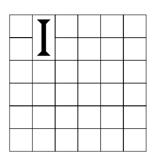
If you did the exercise then you probably made some observations and learned a few tactical tricks in DOMINEERING. One observation is that after a number of dominoes have been placed the board *falls apart* into disconnected regions of empty squares. When you make a move you need to decide what region to play in and how. Suppose that you are the vertical player and that there are two regions of the form:

 and 

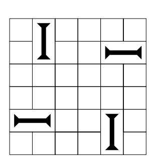
Obviously you could move in either region. However, if you move in the hook-shaped region then your opponent will move in the square. You will have no more moves left so you will lose. If instead you move in the square, then your opponent's only remaining move is in the hook. Now you still have a move in the square to make, and so your opponent will lose. If you are L and your opponent is R play should proceed



This is also why an opening move such as



is good since it *reserves* the two squares in the upper left for you later. In fact, if you play seriously for a while it is quite possible that the board after the first four moves will look something like:



Simply put, the aim of combinatorial game theory is to understand in a more detailed way the principles underlying the sort of observations we have just made about DOMINEERING. We will learn about games in general and how to understand them but, as a bonus, how to play them well!

0.1 Basic Terminology

In this section we will provide an informal introduction to some of the basic concepts and terminology that will be used and a description of how combinatorial games differ from some other types of games.

Combinatorial games

In a *combinatorial game* there are two players who take turns moving alternately. Play continues until the player whose turn it is to move has no legal moves available. No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the details of the game position (or game state) at all times. In this text, the rules of each game we study will ensure that it will end after a finite sequence of moves, and the winner is often determined on the basis of who made the last move. Under *normal play* the last player to move wins. In *misère* play the last player loses.

In fact, combinatorial game theory can be used to analyze some games that do not quite fit the above description. For instance, in DOTS & BOXES, players may make two moves in a row. Most CHECKERS positions are *loopy* and can lead to infinitely long sequences of moves. In GO and CHESS the last mover does not determine the winner. Nonetheless, combinatorial game theory has been applied to analyze positions in each of these games.

By contrast, the classical mathematical theory of games is concerned with *economic games.* In such games the players often play simultaneously and the outcome is determined by a payoff matrix. Each player's objective is to guarantee the best possible payoff against any strategy of the opponent. For a taste of economic game theory, see Problem 5.

The challenge in analyzing economic games stems from simultaneous decisions: each player must decide on a move without knowing the move choice(s) of her opponent(s). The challenge of combinatorial games stems from the sheer quantity of possible move sequences available from a given position.

Combinatorial game theory is most straightforward when we restrict our attention to *short games.* In the play of a short game, a position may never be repeated, and there are only a finite number of other positions that can be reached. We implicity (and sometimes explicitly) assume all games are short in this text.

Introducing the players

The two players of a combinatorial game are traditionally called *Left* (or just L) and *Right* (R). Various conventional rules will help you to recognize who is playing, even without a program:

Left	**Right**
Louise	Richard
Positive	Negative
bLack	White
bLue	Red
Vertical	Horizontal
Female	Male
Green	
Gray	

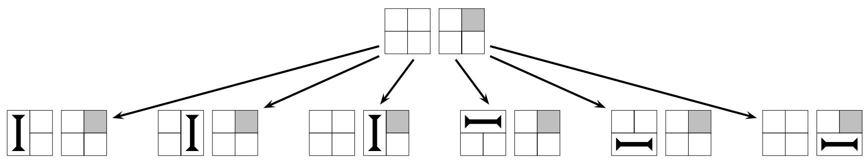
Alice and *Bob* will also make an appearance when the first player is important. To help remember all these conventions, note that despite the fact that they were introduced as long ago as the early 1980s in *WW* [BCG01], the chosen dichotomies reflect a relatively modern "politically correct" viewpoint.

Often, particularly in games involving pieces or in pen and paper games we will need a neutral color. If the game is between blue and red then this neutral color is green (because green is good for everyone!) while if it is between black and white then the neutral color is gray (because gray is neither black nor white!). Of course, this book is printed in black and white, so blue becomes black, red becomes white, and green becomes gray. That is,

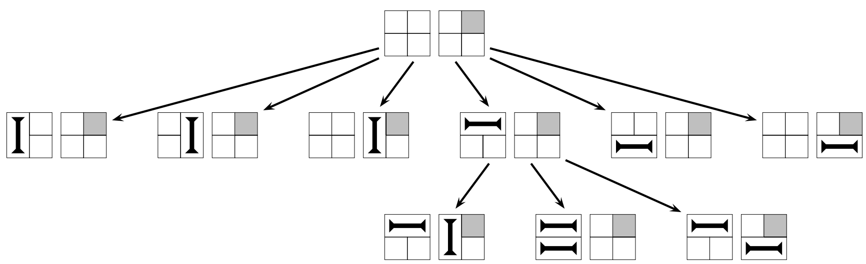
$$\begin{aligned} \text{blue} &= \text{black}, \\ \text{red} &= \text{white}, \\ \text{green} &= \text{gray}. \end{aligned}$$

Options

If a position in a combinatorial game is given and it happens to be Left's turn to move she will have the opportunity to choose from a certain set of moves determined by the rules of the game. For instance in DOMINEERING, where Left plays the vertical dominoes, she may place such a domino on any pair of vertically adjacent empty squares. The positions that arise from exercising these choices are called the *left options* of the original position. Similarly, the *right options* of a position are those which can arise after a move made by Right. The *options* of a position are simply the elements of the union of these two sets. We can draw a game tree of a position by diagrammatically listing its left and right options, with left options appearing below and to the left of the game:

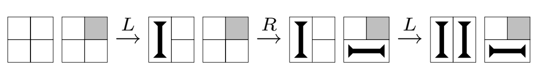


We can show as many or as few *game trees* of options as we wish:

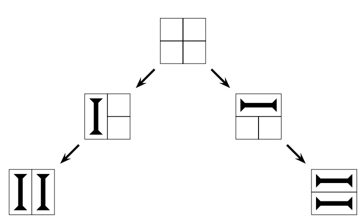


It may seem odd that we are showing two consecutive right moves in a game tree, but much of the theory of combinatorial games is based on analyzing situations where games *decompose* into several subgames. It may well be the case that in some of the subgames of such a decomposition the players do not alternate moves.

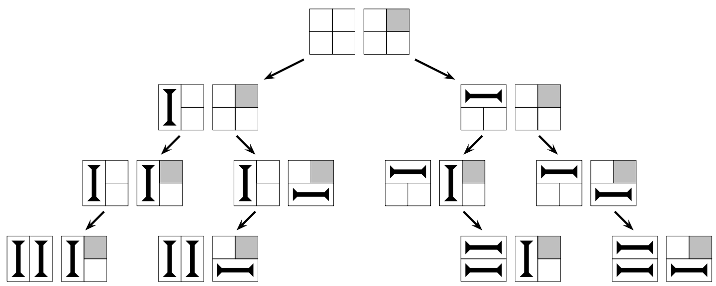
We saw this already in the DOMINEERING "square and hook" example. Left, if she wants to win, winds up making two moves in a row in the square:



Thus, we show the game tree for a square with Left and/or Right moving twice in a row:



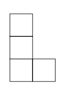
As we will see later, *dominated options* are often omitted from the game tree, when an option shown is at least as good:



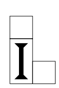
In some games the left options and the right options of a position are always the same. Such games are called *impartial*. The study of impartial combinatorial games is the oldest part of combinatorial game theory and dates back to the early twentieth century. On the other hand the more general study of non-impartial games was pioneered by John H. Conway in *ONAG* [Con01] and by Elwyn Berlekamp, John H. Conway, and Richard K. Guy in *WW* [BCG01]. Since "non-impartial" hardly trips off the tongue, and "partial" has a rather ambiguous interpretation it has become commonplace to refer to non-impartial games as *partizan games.*

To illustrate the difference between these concepts, consider a variation of DOMINEERING called CRAM. CRAM is just like DOMINEERING except that each player can play a domino in either orientation. Thus, it becomes an impartial game since there is now no distinction between legal moves for one player and legal moves for the other.

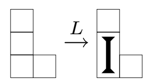
Let's look at a position in which there are only four remaining vacant squares in the shape of an L:



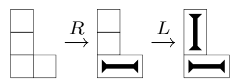
In CRAM the next player to play can force a win by playing a vertical domino at the bottom of the vertical strip, leaving



which contains only two non-adjacent empty squares and hence allows no further moves. In DOMINEERING if Left (playing vertically) is the next player, she can win in exactly this way. However, if Right is the next player his only legal move is to cover the two horizontally adjacent squares, which still leaves a move available to Left. So (assuming solid play) Left will win regardless of who plays first:



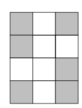
or



Much of the theory that we will discuss is devoted to finding methods to determine who will win a combinatorial game assuming sensible play by both sides. In fact, the eventual loser has no really *sensible* play[1] so a *winning strategy* in a combinatorial game is one that will guarantee a win for the player employing it no matter how his or her opponent chooses to play. Of course such a strategy is allowed to take into account the choices actually made by the opponent — to demand a uniform strategy would be far too restrictive!

Problems

1. Consider the position:



   (a) Draw the complete game trees for both CRAM and DOMINEERING. The leaves (bottoms) of the tree should all be positions in which neither player can move. If two left (or right) options are symmetrically identical, you may omit one.

   (b) Who wins at DOMINEERING if Vertical plays first? Who wins if Horizontal plays first? Who wins at CRAM?

2. Suppose that you play DOMINEERING (or CRAM) on *two* 8×8 chessboards. At your turn you can move on either chessboard (but not both!). Show that the second player can win.

3. Take the ace through five of a suit from a deck of cards and place them face up on the table. Play a game with these as follows. Players alternately pick a card and add it to the righthand end of a row. If the row ever contains a sequence of three cards in increasing order of rank (ace is low), or in decreasing order of rank, then the game ends and the player who formed that sequence is the winner. Note that the sequence need not be

[1]Unless he has some ulterior motive not directly related to the game such as trying to make it last as long as possible so that the bar closes before he has to buy the next round of drinks.

consecutive either in position or value, so for instance, if the play goes $4, 5, 2, 1$ then the $4, 2, 1$ is a decreasing sequence.

(a) Show that this is a proper combinatorial game (the main issue is to show that draws are impossible).

(b) Show that the first player can always win.

4. Start with a heap of counters. As a move from a heap of n counters, you may either:

   - assuming n is not a power of 2, remove the largest power of 2 less than n; or
   - assuming n is even, remove half the counters.

   Under normal play, who wins? How about misère play?

5. The goal of this problem is to give the reader a taste of what is *not* covered in this book. Two players play a 2×2 *zero-sum matrix game.* (Zero sum means that whatever one person loses, the other gains.) The players are shown a 2×2 matrix of positive numbers. Player A chooses a row of the matrix, and player B simultaneously chooses a column. Their choice determines one matrix entry, that being the number of dollars B must pay A. For example, suppose the matrix is

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

   If player A chooses the first row with probability $1/4$, then no matter what player B's strategy is, player A is *guaranteed* to get an average of \$2.50. If, on the other hand, player B chooses the columns with 50-50 odds, then no matter what player A does, player B is *guaranteed* to have to pay an average of \$2.50. Further, neither player can guarantee a better outcome, and so B should pay player A the fair price of \$2.50 to play this game.

   In general, if the entries of the matrix game are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

   as a function of a, b, c, and d, what is the fair price which B should pay A to play? (Your answer will have several cases.)